Review

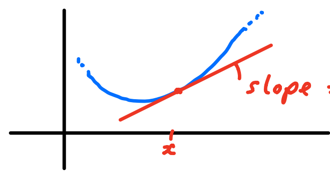
f - function

Derivative of f evaluated at x

Difference Quotient

$$f'(x) = \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{Slope of tangent line at } (x, f(x))$$

$$= \text{Instantaneous rate of change at } x$$



Example

$f(x) = \sqrt{x} \quad (x > 0)$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Laws of Derivatives :

Sum / Difference : $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

Constant Multiple : $\frac{d}{dx}(k f(x)) = k \frac{d}{dx}(f(x))$

Product : $\frac{d}{dx}(f(x) g(x)) = f'(x) g(x) + f(x) g'(x)$

Quotient : $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) g(x) - f(x) g'(x)}{(g(x))^2}$

Chain : $y = f(u)$, $u = g(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) g'(x)$$

Core Examples :

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}(a^x) = \ln(a) a^x$$

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a) x}$$

$$\frac{d}{dx}\left(a^{(g(x))}\right) = \ln(a) a^{(g(x))} \cdot g'(x)$$

$$\frac{d}{dx}(\log_a |g(x)|) = \frac{g'(x)}{\ln(a) g(x)}$$

Example

$$\frac{d}{dx}\left(2^x \log_3(2x+3)\right) = \frac{d}{dx}(2^x) \log_3(2x+3) + 2^x \frac{d}{dx}(\log_3(2x+3))$$

$$= \ln(2) 2^x \log_3(2x+3) + 2^x \cdot \frac{2}{\ln(3)(2x+3)}$$

$F'(x) = f(x) \Rightarrow F$ an antiderivative of f.

Indefinite Integral

arbitrary constant

$\int f(x)\,dx$ = General Antiderivative of $f(x)$ = $F(x) + C$

Core Examples

$$\int x^r\,dx = \begin{cases} \frac{1}{1+r} x^{r+1} + C & \text{if } r \neq -1 \\ \ln|x| + C & \text{if } r = -1 \end{cases}$$

$$\int a^x\,dx = \frac{1}{\ln(a)} a^x + C \Rightarrow \int e^{kx}\,dx = \frac{1}{k} e^{kx} + C$$

Laws of Indefinite Integrals :

Sum / Difference : $\int f(x)\,dx = F(x)+C$, $\int g(x)\,dx = G(x)+C$ $\Rightarrow \int f(x) \pm g(x)\,dx = F(x) \pm G(x) + C$

Constant Multiple : $\int f(x)\,dx = F(x)+C = \int k f(x)\,dx = kF(x)+C$

Substitution Rule : $\int f(g(x))\,g'(x) = F(g(x)) + C$

x - world (Original variable)

$$\left(\begin{aligned} u &= g(x) \\ \Rightarrow \frac{du}{dx} &= g'(x) \\ \Rightarrow dx &= \frac{du}{g'(x)} \end{aligned} \right)$$

u - world (New variable)

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C \xleftarrow{\text{Replace } u \text{ with } g(x)} F(u) + C \xleftarrow{\text{Calculate integral}} \int f(u)\,du$$

Replace dx

Replace $g(x)$ with u

$$\int f(g(x))\,\cancel{g'(x)}\,\frac{du}{\cancel{g'(x)}} = \int f(g(x))\,du$$

Important Examples

1/ $\int 2x+1\,(x^2+x+3)^{1/2}\,dx \rightsquigarrow u = x^2+x+3$

2/ $\int x e^{(x^2)}\,dx \;/\; \int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx \;/\; \int \frac{3^{1/x}}{x^2}\,dx \rightsquigarrow u = x^2 / \sqrt{x} / {}^1/_x$

3/ $\int \frac{(\ln(x))^3}{2x}\,dx \rightsquigarrow u = \ln(x)$

4/ $\int (ax+b)(cx+d)^5\,dx \rightsquigarrow u = cx+d$

E.g. $\int (x+1)\sqrt{x+2}\ dx$

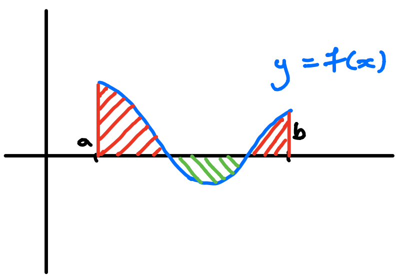
$u = x+2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow dx = du \Rightarrow$

$(\Rightarrow x = u-2)$

$$\int (x+1)\sqrt{x+2}\ dx = \int (x+1)\sqrt{x+2}\ du = \int (x+1)\sqrt{u}\ du$$

$$= \int (u-1)\sqrt{u}\, du = \int u^{3/2} - u^{1/2}\, du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{2}{3}(x+2)^{3/2} + C$$



$$\Rightarrow \int_a^b f(x)dx = \text{Area}(\ ) - \text{Area}(\ )$$

$= \underline{\text{Net Area}}$ enclosed by $y = f(x)$ and x-axis between a and b

Fundamental Theorem of Calculus :

Notation

$$F'(x) = f(x) \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Remark In general $F(b) \neq \text{Area}(\ )$ and $F(a) \neq \text{Area}(\ )$

Example $\int_0^1 x e^{(x^2)}\, dx = ?$

$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Rightarrow$

$$\int x e^{(x^2)} dx = \int \frac{1}{2} e^u\, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{(x^2)} + C$$

$$\Rightarrow \int_0^1 x e^{(x^2)} dx = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{1}{2}e - \frac{1}{2}$$

$f(x) \geq 0$ on $[a,b] \Rightarrow \displaystyle\int_a^b f(x)\,dx = \text{Area}(\;)$

$f(x) \leq 0$ on $[a,b] \Rightarrow \displaystyle\int_a^b f(x)\,dx = -\text{Area}(\;) \Rightarrow \text{Area}(\;) = -\int_a^b f(x)\,dx$

Total Area enclosed by $y=f(x)$ and x-axis between a and b $= \text{Area}(\;) + \text{Area}(\;)$

Must be calculated separately using different definite integrals

Example: $v(t)$ = speed at time t.

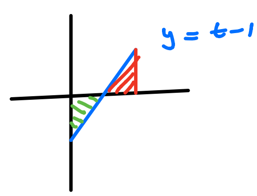
Distance travelled in positive direction

Distance travelled in negative direction

$\displaystyle\int_a^b v(t)\,dt$ = Net Distance travelled between $t=a$ and $t=b$ $= \text{Area}(\;) - \text{Area}(\;)$

Total Distance travelled between $t=a$ and $t=b$ $= \text{Area}(\;) + \text{Area}(\;)$

$a=0$, $b=2$, $v(t) = t-1$



Net Distance travelled between $t=0$ and $t=2$ $= \displaystyle\int_0^2 t-1\,dt = \frac{1}{2}t^2 - t\Big|_0^2 = 0$

Total Distance travelled between $t=0$ and $t=2$ $= \left(\displaystyle\int_1^2 t-1\,dt\right) + \left(-\int_0^1 t-1\,dt\right)$

$$= \left(\frac{1}{2}t^2 - t\Big|_1^2\right) + \left(t - \frac{1}{2}t^2\Big|_0^1\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Sign Analysis : Finding when $g(x) > 0$ or $g(x) < 0$

1/ Find c such that

A/ $g(c) = 0$ **or** B/ $g(c)$ DNE (discontinuous really)

2/ Draw a number line and mark these points.

3/ Choose a test point between each and evaluate to see if $+$ or $-$.

Finding where f increasing / decreasing :

1/ Calculate $f'(x)$

2/ Do sign analysis on $f'(x)$

3/ $f'(x) > 0 \Rightarrow f$ increasing

$f'(x) < 0 \Rightarrow f$ decreasing

Finding relative extrema :

1/ Calculate $f'(x)$

2/ Do sign analysis on $f'(x)$

3/

$f(c)$ exists

$+ \quad c \quad -$ $\quad f'(x) \Rightarrow f(c)$ rel. max

$f(c)$ exists

$- \quad c \quad +$ $\quad f'(x) \Rightarrow f(c)$ rel. min

Finding where f is concave up and concave down :

1/ Calculate $f''(x)$

2/ Do sign analysis on $f''(x)$

3/ $f''(x) > 0 \Rightarrow$ Concave Up

$f''(x) < 0 \Rightarrow$ Concave Down

Finding Inflection Points

1/ Calculate $f''(x)$

2/ Do sign analysis on $f''(x)$.

3/

$f'(c)$ exists

$+ \quad c \quad -$ $\quad f''(x) \quad \Rightarrow (c, f(c))$ inflection

$f'(c)$ exists

$- \quad c \quad +$ $\quad f''(x) \quad \Rightarrow (c, f(c))$ inflection

Finding absolute extrema on closed interval (f continuous on $[a,b]$)

1/ Calculate $f'(x)$

2/ Find points c such that A/ or B/ hold for $f'(x)$.

3/ Evaluate $f(c)$ for such points and compare

Finding absolute extrema on non-closed interval

(f continuous on interval)

1/ Calculate $f'(x)$

2/ Do sign analysis on f'.

3/ If c is only critical number inside interval

$f'(x)$: $+$ c $-$ $\Rightarrow$ $f(c)$ absolute max

$f'(x)$: $-$ c $+$ $\Rightarrow$ $f(c)$ absolute min

4/ If more than one critical number we need a little luck.

E.g. $-$ a $+$ b $-$ c $+$ $\Rightarrow$ Either $f(a)$ or $f(c)$ is absolute min

Putting It All Together : Curve Sketching

Strategy:

1/ Determine domain (look for square roots and denominators being 0)

2/ If possible find x and y intercepts.

3/ a) Look for horizontal asymptotes, ie. $\lim_{x \to \pm\infty} f(x) = L \Rightarrow y = L$ a horizontal asymptote

Important examples : Rational and exponential functions

b) Look for vertical asymptotes, ie $\lim_{x \to a^{+/-}} f(x) = \infty \Rightarrow x = a$ vertical asymptotes

Important examples : Rational and logarithmic functions.

$x = 0$ for both $\frac{1}{x}$ and $\ln|x|$

4/ Is function odd/even ?

5/ Determine increasing/decreasing/relative extrema using $f'(x)$.

6/ Determine concavity/inflection points using $f''(x)$

7/ Plot intercepts, critical points, inflection points and asymptotes.

Connect together following sign analysis done in 5/ and 6/

Example Sketch $y = \dfrac{2x+2}{x-1}$ $\left(= f(x)\right)$

1/ Domain = All $x \neq 1$

2/ $f(0) = \dfrac{2}{-1} = -2 = y$ - intercept

$f(x) = 0 \Rightarrow 2x+2 = 0 \Rightarrow x = -1 = x$ - intercept

3/ A/ $\displaystyle\lim_{x \to \pm\infty} \frac{2x+2}{x-1} = 2 \Rightarrow y = 2$ horizontal asymptote in both directions.

B/ $\underbrace{(x-1)}_{\text{denominator}} = 0 \Rightarrow x = 1$, $\overbrace{2 \cdot 1 + 2}^{\text{numerator}} = 4 \neq 0$

$\Rightarrow x = 1$ vertical asymptote

4/ Neither odd or even

5/ $f(x) = \dfrac{u}{v}$, $u = 2x+2$, $v = x-1$

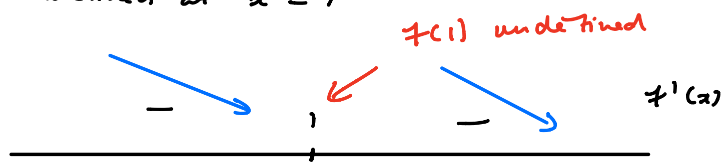
$\Rightarrow u'(x) = 2$, $v'(x) = 1$

$$\Rightarrow f'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{(v(x))^2} = \frac{2(x-1) - (2x+2) \cdot 1}{(x-1)^2}$$

$$= \frac{-4}{(x-1)^2}$$

A/ $f'(x) = 0 \Rightarrow \dfrac{-4}{(x-1)^2} = 0$ (No solutions)

B/ f' undefined at $x = 1$



$f'(0) = \dfrac{-4}{(-1)^2} < 0$ $\qquad$ $f'(2) = \dfrac{-4}{(2-1)^2} < 0$

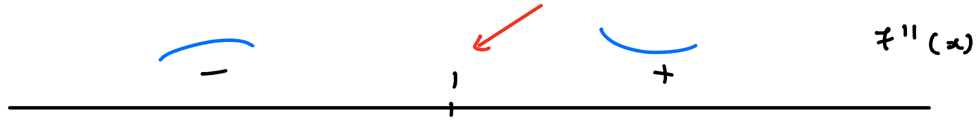
6/ $$f''(x) = \frac{d}{dx}\left(\frac{-4}{(x-1)^2}\right) = -4\,\frac{d}{dx}\left((x-1)^{-2}\right)$$ Chain Rule

$$= (-4) \cdot (-2)\,(x-1)^{-3}$$

$$= \frac{8}{(x-1)^3}$$

A/ $f''(x) = 0 \Rightarrow \dfrac{8}{(x-1)^3} = 0$ (No solutions)

B/ f'' undefined at $x = 1$ $f'(1)$ undefined $\Rightarrow$ Not inflection

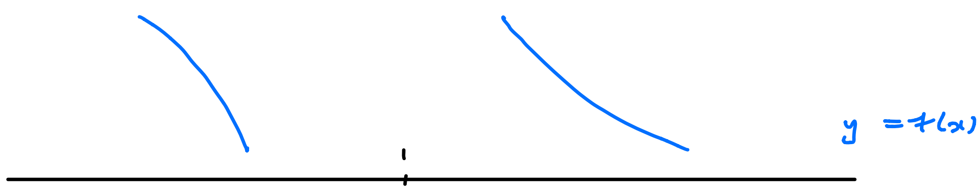


$f''(0) = -8 < 0$ $f''(2) = 8 > 0$

7/ No critical points or inflections.

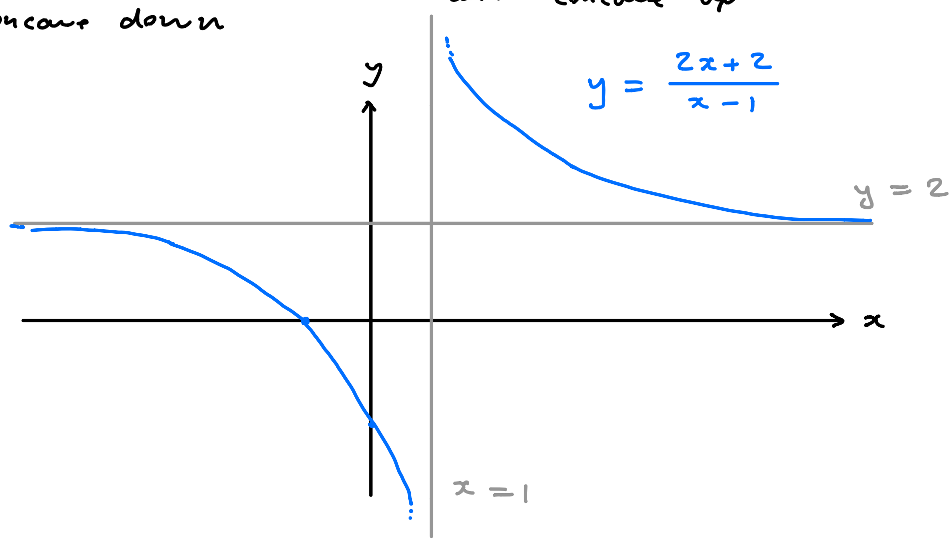
x-intercept : $(-1, 0)$

y-intercept : $(0, -2)$



f decreasing and concave down

f decreasing and concave up



Constrained Optimization :

1/ Identify objective.

2/ Draw picture, label unknowns

3/ Write objective in terms of unknowns. Gives Objective Formula

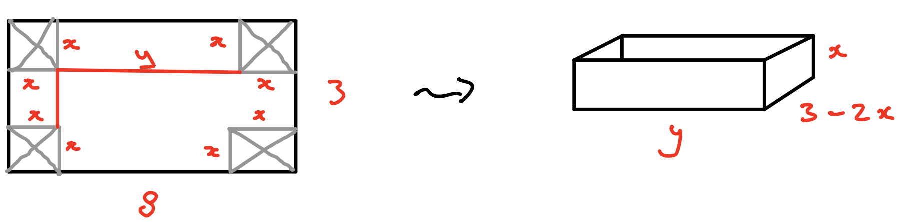
4/ Identify constraint. Express it as Constraint Equation in terms of unknowns.

5/ Solve constraint equation in on unknown and sub into objective. Gives f, a single variable function.

6/ Find appropriate domain by looking at f, the picture and the constraint. Find absolute max / min.

Example What is max volume of a box formed by cutting square corners from a 3×8 foot piece of card board?

Objective : Maximize Volume



$$\text{Volume} = y\,x\,(3-2x)$$

Constraint : $x + y + x = 8 \Rightarrow y = 8 - 2x$

$$\Rightarrow y\,x\,(3-2x) = (8-2x)\,x\,(3-2x) = 4x^3 - 22x^2 + 24x = f(x)$$

Domain : $x \geq 0 \quad 2x \leq 3 \Rightarrow x \leq \frac{3}{2}$, $[0, \frac{3}{2}]$

$f'(x) = 12x^2 - 44x + 24 = 4(3x - 11x + 6)$

$= 4(3x-2)(x-2)$

A/ $f'(x) = 0 \Rightarrow x = 2/3$ or 2

B/ f' continuous on $[0, 3/2]$

$\Rightarrow$ Critical numbers in $[0, 3/2]$ are $0, 3/2, 2/3$

$f(0) = 0$

$f(3/2) = 0$

$f(2/3) = (8 - \frac{4}{3}) \frac{2}{3} \cdot (3 - \frac{4}{3}) > 0$

$\Rightarrow \frac{20}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} = \frac{200}{3}$ is max volume.

Remark : Make sure you know volume/surface area of cylinders / rectangular boxes.

Implicit Differentiation : Finding $\frac{dy}{dx}$ when we don't have formula for y, just an equation in x and y.

Strategy : Differentiate both sides with respect to x.
Expand until only terms are $x, y, \frac{dy}{dx}$.
Solve for $\frac{dy}{dx}$.

<u>Example</u> Calculate slope of tangent to $x^2y^3=1$ when $x=1$.

$x=1 \Rightarrow 1^2y^3=1 \Rightarrow y^3=1 \Rightarrow y=1$

$\Rightarrow (1,1)$ is relevant point on curve.

$$\frac{d}{dx}(x^2y^3) = \frac{d}{dx}(1)$$

$y=f(x) \Rightarrow \frac{d}{dx}(y^3) = \frac{d}{dx}(f(x))^3$

$= 3(f(x))^2 f'(x)$

$= 3y^2\frac{dy}{dx}$

??

$$\Rightarrow \frac{d}{dx}(x^2)\,y^3 + x^2\frac{d}{dx}(y^3) = 0$$

$$\Rightarrow 2xy^3 + x^2\cdot 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xy^3}{3x^2y^2} = \frac{-2y}{3x}$$

$\Rightarrow$ Slope of tangent at $(1,1) = \frac{-2\cdot 1}{3\cdot 1} = \frac{-2}{3}$.

<u>Remark</u>: Key observation is $\frac{d}{dx}(g(y)) = g'(y)\frac{dy}{dx}$

Economics Applications

$R(x), C(x), P(x)$ = Revenue, cost, profit from making/selling x units of a product

$R'(x), C'(x), P'(x)$ = Marginal revenue, cost, profit

$$P(x) = R(x) - C(x) \implies P'(x) = R'(x) - C'(x)$$

$$P(c) = R(c) - C(c) = 0 \implies x = c$$ break-even quantity

Supply and Demand

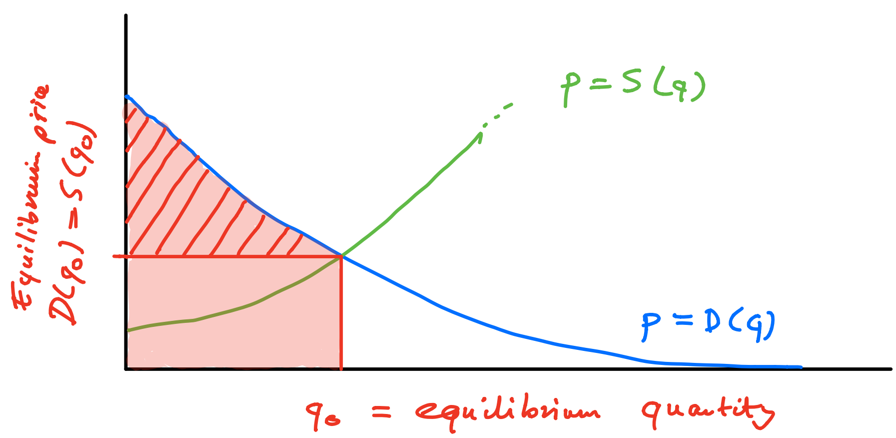
q = number of units sold / supplied

p = price per unit

q independent variable

Demand Equation : $p = D(q)$ (Decreasing)

Supply Equation : $p = S(q)$ (Increasing)



$$\text{Area}(\;) = \int_0^{q_0} D(q)\,dq =$$ Total consumers are willing to pay for q_0 units

constant

$$\text{Area}(\;) = \int_0^{q} D(q) - D(q_0)\,dq = \underline{\text{Consumer Surplus}}$$

Elasticity of Demand

$$\text{Elasticity} = \frac{-p}{q} \cdot \frac{dq}{dp}$$

Rate of change of price with respect to price

Elasticity < 1 ⇒ Inelastic demand ⇒ Should increase price to raise revenue

Elasticity > 1 ⇒ Elastic demand ⇒ Should decrease price to raise revenue

Elasticity = 1 ⇒ Unit Elasticity ⇒ Potential max revenue

Straightforward Example : $q = 20 - p^2$ $(\Rightarrow p^2 = 20 - q)$

$$\Rightarrow \frac{dq}{dp} = -2p \Rightarrow E = \frac{-p}{q} \cdot (-2p) = \frac{2p^2}{q}$$

$$E(p) = \frac{2p^2}{20-p^2}, \quad E(q) = \frac{40-2q}{q}$$

Same quantity different independent variable.

Hard Example : $p^2 + 2pq + q^2 = 4$

Can't easily solve in q.

Must use implicit differentiation.

$$\frac{d}{dp}(p^2 + 2pq + q^2) = \frac{d}{dp}(4)$$

$$\Rightarrow 2p + 2q + 2p\frac{dq}{dp} + 2q\frac{dq}{dp} = 0$$

$$\Rightarrow \quad (2p + 2q)\frac{dq}{dp} = -2p - 2q$$

$$\Rightarrow \quad \frac{dq}{dp} = \frac{-2p - 2q}{2p + 2q} = -1$$

constant

$$\Rightarrow \quad E = \frac{-p}{q}(-1) = \frac{p}{q}$$

Must leave in terms of both p and q.